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SOME TOPICS ON MULTIVALENT FUNCTIONS

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I. INTRODUCTION

Let $\mathcal{A}(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1,2,3,\ldots\}; n \in \mathbb{N}) \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z: |z| < 1\}$.

A function $f(z)$ belonging to the class $\mathcal{A}(p,n)$ is said to be p-valently starlike of order α if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for some α $(0 \leqq \alpha < p)$ and for all $z \in \mathbb{U}$. We denote by $\mathcal{S}(p,n,\alpha)$ the subclass of $\mathcal{A}(p,n)$ consisting of functions which are p-valently starlike of order α.

A function $f(z)$ in the class $\mathcal{A}(p,n)$ is said to be p-valently convex of order α if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.3}$$

for some α $(0 \leqq \alpha < p)$ and for all $z \in \mathbb{U}$. We denote by $\mathcal{K}(p,n,\alpha)$ the subclass of $\mathcal{A}(p,n)$ consisting of all such functions. Note that $f(z) \in \mathcal{K}(p,n,\alpha)$ if and only if $zf'(z) \in \mathcal{S}(p,n,\alpha)$.

Further, a function $f(z)$ belonging to the class $\mathcal{A}(p,n)$ is said to be p-valently close-to-convex of order α if there exists a function $g(z) \in \mathcal{K}(p,n,0)$ such that

$$(1.4)\qquad \mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha$$

for some α ($0 \leqq \alpha < 1$) and for all $z \in \mathcal{U}$. Also we denote by $\mathcal{C}(p,n,\alpha)$ the subclass of $\mathcal{A}(p,n)$ consisting of functions which are p-valently close-to-convex of order α.

2. p-VALENTLY α-CONVEX FUNCTIONS OF ORDER β

A function $f(z)$ in the class $\mathcal{A}(p,n)$ is said to be p-valently α-convex of order β if it satisfies

$$(2.1)\qquad \mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta$$

for some α ($\alpha \geqq 0$), β ($0 \leqq \beta < p$), and for all $z \in \mathcal{U}$. Denoting by $\mathcal{M}(p,n,\alpha,\beta)$ the subclass of $\mathcal{A}(p,n)$ consisting of all such functions, we see that $\mathcal{M}(p,n,0,\beta) = \mathcal{S}(p,n,\beta)$ and $\mathcal{M}(p,n,1,\beta) = \mathcal{K}(p,n,\beta)$.

In particular, $\mathcal{M}(1,1,\alpha,\beta)$ when $p = 1$ and $n = 1$ is the class which was studied by Zmorovich and Pokhilevich [6], and $\mathcal{M}(p,1,\alpha,0)$ when $n = 1$ and $\beta = 0$ is the class which was studied by Owa and Ren [4], and Ren and Owa [5].

LEMMA 2.1 (Miller and Mocanu [1]). Let $\phi(u,v)$ be a complex valued function,

$$\phi: \mathcal{D} \longrightarrow \mathcal{C}, \quad \mathcal{D} \subset \mathcal{C}^2 \qquad (\mathcal{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in $\mathcal{D}$;

(ii) $(1,0) \in \mathcal{D}$ and $\mathrm{Re}\{\phi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\phi(iu_2,v_1)\} \leqq 0$ for all $(iu_2,v_1) \in \mathcal{D}$ such that $v_1 \leqq -n(1+u_2^2)/2$.

Let $q(z) = 1 + q_n z^n + q_{n+1}z^{n+1} + \dots$ be regular in the unit disk $\mathbb{U}$ such that $(q(z), zq'(z)) \in \mathbb{D}$ for all $z \in \mathbb{U}$. If

$$\mathrm{Re}\{\phi(q(z), zq'(z))\} > 0 \qquad (z \in \mathbb{U}),$$

then

$$\mathrm{Re}\{q(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Using the above lemma, we derive

THEOREM 2.1. If $f(z) \in \mathcal{M}(p,n,\alpha,\beta)$ with $0 \leqq (p-\alpha n)/2 \leqq \beta < p$, then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma(p,n,\alpha,\beta) \qquad (z \in \mathbb{U}), \tag{2.2}$$

where

$$\gamma(p,n,\alpha,\beta) = \frac{2\beta - \alpha n + \sqrt{(2\beta-\alpha n)^2 + 8\alpha p n}}{4}. \tag{2.3}$$

Therefore, $f(z)$ is in the class $\mathcal{S}(p,n,\gamma(p,n,\alpha,\beta))$.

PROOF. Define the function $q(z)$ by

$$\frac{zf'(z)}{f(z)} = p(\gamma_1 + (1-\gamma_1)q(z)) \tag{2.4}$$

with $\gamma_1 = \gamma(p,n,\alpha,\beta)/p$. Then $q(z) = 1 + q_n z^n + q_{n+1}z^{n+1} + \ldots$ is regular in the unit disk $\mathbb{U}$. Making use of the logarithmic differentiations of both sides in (2.4), we obtain

$$1 + \frac{zf''(z)}{f'(z)} = p(\gamma_1 + (1-\gamma_1)q(z)) + \frac{(1-\gamma_1)zq'(z)}{\gamma_1 + (1-\gamma_1)q(z)}. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta\right\} \tag{2.6}$$

$$= \mathrm{Re}\left\{p\gamma_1 - \beta + p(1 - \gamma_1)q(z) + \frac{\alpha(1 - \gamma_1)zq'(z)}{\gamma_1 + (1 - \gamma_1)q(z)}\right\}$$

$> 0.$

Letting $u = u_1 + iu_2$, $v = v_1 + iv_2$, and

$$\phi(u,v) = p\gamma_1 - \beta + p(1 - \gamma_1)u + \frac{\alpha(1 - \gamma_1)v}{\gamma_1 + (1 - \gamma_1)u}, \tag{2.7}$$

we know that

(i) $\phi(u,v)$ is continuous in $\mathbb{D} = \left(\mathbb{C} - \{\frac{\gamma_1}{\gamma_1 - 1}\}\right) \times \mathbb{C}$;

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\{\phi(1,0)\} = p - \beta > 0$;

(iii) for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leqq -n(1 + u_2^2)/2$,

$$\begin{aligned} \mathrm{Re}\{\phi(iu_2, v_1)\} &= p\gamma_1 - \beta + \frac{\alpha\gamma_1(1 - \gamma_1)v_1}{\gamma_1^2 + (1 - \gamma_1)^2u_2^2} \\ &\leqq p\gamma_1 - \beta - \frac{\alpha\gamma_1(1 - \gamma_1)n(1 + u_2^2)}{2\{\gamma_1^2 + (1 - \gamma_1)^2u_2^2\}} \\ &\leqq 0 \end{aligned}$$

because $0 \leqq (p - \alpha n)/2 \leqq \beta < p$ and $0 < \gamma_1 < 1$.

This implies that the function $\phi(u,v)$ satisfies the conditions in Lemma 2.1. Thus, applying Lemma 2.1, we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > p\gamma_1 = \gamma(p,n,\alpha,\beta) \qquad (z \in \mathbb{U}), \tag{2.8}$$

which completes the proof of Theorem 2.1.

Making $p = 1$, Theorem 2.1 leads to

COROLLARY 2.1. If $f(z) \in \mathcal{M}(1,n,\alpha,\beta)$ with $0 \leqq (1-\alpha n)/2 \leqq \beta < 1$, then

$$\text{(2.9)} \qquad \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{2\beta - \alpha n + \sqrt{(2\beta - \alpha n)^2 + 8\alpha n}}{4} \qquad (z \in \mathcal{U}).$$

Taking $\alpha = 1$ in Theorem 2.1, we have

COROLLARY 2.2. If $f(z) \in \mathcal{K}(p,n,\beta)$ with $0 \leqq (p-n)/2 \leqq \beta < p$, then

$$\text{(2.10)} \qquad \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{2\beta - n + \sqrt{(2\beta - n)^2 + 8pn}}{4} \qquad (z \in \mathcal{U}).$$

Letting $\beta = \alpha n/2$, we have

COROLLARY 2.3. If $f(z) \in \mathcal{M}(p,n,\alpha,\alpha n/2)$ with $\alpha n > 2p$, then

$$\text{(2.11)} \qquad \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \sqrt{\frac{\alpha p n}{2}} \qquad (z \in \mathcal{U}).$$

Further, using the same technique as in the proof of Theorem 2.1, we prove

THEOREM 2.2. If $f(z) \in \mathcal{A}(p,n)$ satisfies

$$\text{(2.12)} \qquad \operatorname{Re}\left\{ (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \beta \qquad (z \in \mathcal{U})$$

for some α ($\alpha \geqq 0$) and β ($\beta > p$), then

$$\text{(2.13)} \qquad \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \gamma(p,n,\alpha,\beta) \qquad (z \in \mathcal{U}),$$

where $\gamma(p,n,\alpha,\beta)$ is given by (2.3).

3. p-VALENTLY CLOSE-TO-CONVEX OF ORDER δ

In order to derive our next result, we need the following lemma.

LEMMA 3.1. If $f(z) \in \mathcal{S}(p,n,\alpha)$, then

$$\operatorname{Re}\left(\frac{f(z)}{z^p}\right)^{\beta} > \frac{n}{2\beta(p-\alpha)+n} \qquad (z \in \mathcal{U}), \tag{3.1}$$

where $0 < \beta \leqq n/2(p-\alpha)$.

Now, we prove

THEOREM 3.1. If $f(z) \in \mathcal{A}(p,n)$ satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha - \beta \qquad (z \in \mathcal{U}) \tag{3.2}$$

for some α $(\alpha > 0)$ and β $(0 < \beta \leqq n/2(1-\gamma))$, where $\gamma = \alpha/(p+\beta)$, then $f(z) \in \mathcal{C}(p,n,\delta(p,n,\alpha,\beta))$, where

$$\delta(p,n,\alpha,\beta) = \frac{n(p+\beta)}{(p+\beta)(n+2\beta)-2\alpha\beta}. \tag{3.3}$$

PROOF. Note that $f(z)$ satisfies

$$\operatorname{Re}\left\{1 + \beta + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U}). \tag{3.4}$$

We define the function $g(z)$ by

$$\frac{zg'(z)}{g(z)} = \frac{1}{p+\beta}\left\{1 + \beta + \frac{zf''(z)}{f'(z)}\right\}. \tag{3.5}$$

Then $g(z)$ belongs to the class $\mathcal{S}(1,n,\gamma)$ with $\gamma = \alpha/(p+\beta)$.

Noting that

$$\frac{zf'(z)}{g(z)^p} = \left(\frac{g(z)}{z}\right)^{\beta} \tag{3.6}$$

and $g(z)^p \in \mathcal{S}(p,n,p\gamma)$, Lemma 3.1 leads to

$$\text{(3.7)} \qquad \mathrm{Re}\left\{ \frac{zf'(z)}{g(z)^p} \right\} = \mathrm{Re}\left(\frac{g(z)}{z} \right)^{\beta}$$

$$> \frac{n}{2\beta(1-\gamma)+n}$$

$$= \delta(p,n,\alpha,\beta).$$

This completes the assertion of Theorem 3.1.

Taking $\alpha = 0$ in Theorem 3.1, we have

COROLLARY 3.1. If $f(z) \in \mathcal{A}(p,n)$ satisfies

$$\text{(3.8)} \qquad \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\beta \qquad (z \in \mathcal{U})$$

for some β $(0 < \beta \leqq n/2)$, then $f(z) \in \mathcal{C}(p,n,\delta(n,\beta))$, where $\delta(n,\beta) = n/(n+2\beta)$.

Further, making $\alpha = \beta$ in Theorem 3.1, we have

COROLLARY 3.2. If $f(z) \in \mathcal{K}(p,n,0)$, then $f(z) \in \mathcal{C}(p,n,1/2)$. Therefore, $\mathcal{K}(p,n,0)$ is the subclass of $\mathcal{C}(p,n,1/2)$.

4. CONVOLUTIONS FOR MULTIVALENT FUNCTIONS

For functions $f_j(z)$, $j = 1,2$, defined by

$$\text{(4.1)} \qquad f_j(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,j} z^k \qquad (p \in \mathbb{N}; n \in \mathbb{N}),$$

we denote by $f_1 * f_2(z)$ the convolution (or Hadamard product) of two functions $f_1(z)$ and $f_2(z)$, that is,

$$f_1 * f_2(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k . \tag{4.2}$$

LEMMA 4.1 (Owa [2]). If $f(z) \in \mathcal{S}(1,1,\alpha)$ and $g(z) \in \mathcal{K}(1,1,\beta)$, then $f*g(z) \in \mathcal{S}(1,1,\alpha)$.

LEMMA 4.2 (Owa [2]). If $f(z) \in \mathcal{K}(1,1,\alpha)$ and $g(z) \in \mathcal{K}(1,1,\beta)$, then $f*g(z) \in \mathcal{K}(1,1,\gamma)$, where $\gamma = \max(\alpha,\beta)$.

LEMMA 4.3 (Owa [3]). If $f(z) \in \mathcal{C}(1,1,\alpha)$ and $g(z) \in \mathcal{K}(1,1,\beta)$, then $f*g(z) \in \mathcal{C}(1,1,\gamma)$, where $\gamma = \max(\alpha,\beta)$.

In view of the above lemmas, we have

REMARK. (i) $f(z) \in \mathcal{S}(1,n,\alpha)$, $g(z) \in \mathcal{K}(1,n,\beta)$

$$\Longrightarrow f*g(z) \in \mathcal{S}(1,n,\alpha).$$

(ii) $f(z) \in \mathcal{K}(1,n,\alpha)$, $g(z) \in \mathcal{K}(1,n,\beta)$

$$\Longrightarrow f*g(z) \in \mathcal{K}(1,n,\gamma), \quad \gamma = \max(\alpha,\beta).$$

(iii) $f(z) \in \mathcal{C}(1,n,\alpha)$, $g(z) \in \mathcal{K}(1,n,\beta)$

$$\Longrightarrow f*g(z) \in \mathcal{C}(1,n,\gamma), \quad \gamma = \max(\alpha,\beta).$$

Finally, from the above remark, we give

CONJECTURE. (i) $f(z) \in \mathcal{S}(p,n,\alpha)$, $g(z) \in \mathcal{K}(p,n,\beta)$

$$\Longrightarrow f*g(z) \in \mathcal{S}(p,n,\alpha).$$

(ii) $f(z) \in \mathcal{K}(p,n,\alpha)$, $g(z) \in \mathcal{K}(p,n,\beta)$

$$\Longrightarrow f*g(z) \in \mathcal{K}(p,n,\gamma), \quad \gamma = \max(\alpha,\beta).$$

(iii) $f(z) \in \mathcal{C}(p,n,\alpha)$, $g(z) \in \mathcal{K}(p,n,\beta)$

$$\Longrightarrow f*g(z) \in \mathcal{C}(p,n,\gamma), \quad \gamma = \max(\alpha,\beta).$$

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